A Note on Contractible Edges in Chordal graphs

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Abstract. Contraction of an edge merges its end points into a new vertex which is adjacent to each neighbor of the end points of the edge. An edge in a k-connected graph is *contractible* if its contraction does not result in a graph of lower connectivity. We characterize contractible edges in chordal graphs using properties of tree decompositions with respect to minimal vertex separators.

1 Introduction

Chordal graphs are also known as triangulated graphs [3] and have applications in the study of linear sparse systems, scheduling and relational database systems. In this paper, we focus on k-connected chordal graphs. We study the impact of contraction on connectivity in k-connected chordal graphs. In a graph G, contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex z. In the resulting graph, the edges incident on u and v are incident on z. Edge contraction and in general clique contraction plays a significant role in the proof of the Perfect Graph Theorem, see [8]. Edge contraction also plays a significant role in min-cut algorithms by using the basic property that the contraction of an edge can only increase the size of the min-cut. The basic idea exploited in randomized algorithms for min-cut is that contracting a randomly chosen edge does not increase the size of the min-cut [9]. This leads to expected polynomial time algorithms for min-cut, and these algorithms are fundamentally different from the classical max-flow based techniques.

1.1 Past Results on Contractible Edges

As with many a problem in Graph Theory, the study of contractible edges was initiated by Tutte in [7] where a constructive characterization of 3-connected graphs was presented. One consequence of this characterization was that in any 3-connected graph with at least five vertices, there is at least one contractible edge. In the work by Saito et. al [6], this lower bound was improved to $\frac{|V(G)|}{2}$, and the structure of graphs that have exactly so many contractible edges is studied. For k-connected graph with $k \geq 4$, it is still ongoing research to find necessary and sufficient conditions for the presence of contractible edges. For example, Thomassen [2] has shown that there is a contractible edge in triangle-free k-connected graphs in which the minimum degree is more than $\frac{3k-3}{2}$. Kriesell's

survey of contractible edges [4] is an excellent source for many results in this area, and is also the motivation point of our work.

1.2 Definitions

We have, to a large extent, followed the notation and definitions as in the Graph Theory text by West[1]. Let $G = (V, E)$ be an undirected non weighted graph where $V(G)$ is the set of vertices and $E(G) \subseteq \{\{u, v\} | u, v \in V(G),\ u \neq v\}$. Order of G and size of G are $|V(G)|$ and $|E(G)|$, respectively. The *neighborhood* of a vertex v in a graph G is the set $\{u | \{u, v\} \in E(G)\}$ and is denoted by $N_G(v)$. A separating set or cut set of a graph G is a set $S \subseteq V(G)$ such that the induced subgraph, denoted by $G - S$, on the vertex set $V(G) \setminus S$ has more than one connected component. The vertex connectivity of a graph G, written $\kappa(G)$, is the minimum cardinality of a vertex set S such that $G - S$ is disconnected or has only one vertex. γ_G is the set of all minimum order cut sets. We let $G.e$ denote the graph obtained by contracting an edge $e = \{u, v\}$ in G such that $V(G.e) = V(G) \setminus \{u, v\} \cup \{z_{uv}\}$ and $E(G.e) = \{\{z_{uv}, x\} | \{u, x\}$ or $\{v, x\} \in E(G)\} \bigcup \{\{x, y\} | x \neq u, y \neq v$ in $E(G)\}$. An edge $e \in E(G)$ is contractible if the connectivity of $G.e$ is same as the connectivity of G. $E_c(G)$ denotes the set of contractible edges in G. A k-connected graph G is said to be contraction critical, if for each edge e, connectivity of $G.e$ is smaller than the connectivity of G. The following lemma relates cut sets and contractible edges [4].

Lemma 1. *An edge $e = \{u, v\}$ of G is non contractible if and only if there is a minimum cut set $T \in \gamma_G$ such that $u \in T$ and $v \in T$.*

A tree decomposition of a graph $G = (V, E)$ is a tree T, where each node x has a label $l(x) \subseteq V(G)$ such that:

- $\bigcup\limits_{x \in V(T)} l(x) = V(G)$.(We say that "all vertices are covered.")
- For any edge $\{v, w\} \in E(G)$, there exists a node x in T such that $v, w \in l(x)$.(We say that "all edges are covered.")
- For any $v \in V(G)$, the set of all nodes of T whose label contains v form a connected subtree in T.(We call this the "connectivity condition")

Chordal Graph Preliminaries

A *chord* of a cycle C is an edge not in C whose endpoints lie in C. A *chordless cycle* in G is a cycle of length at least 4 in G that has no chord. A graph G is *chordal* if it is simple and has no chordless cycle. We can represent a chordal graph G using a tree decomposition T as follows: for each vertex $x \in V(T)$ the associated label $l(x) \subseteq V(G)$ induces a maximal clique in G, and for each $v \in V(G)$, T_v, the subgraph of T induced by the set $\{x \in V(T) | v \in l(x)\}$, is a tree. We use M to denote the set of minimal vertex separators of G, and the graph to which the symbol M applies is always clear from the context. A stable (or independent) set is a set of pairwise nonadjacent vertices of the graph G. A *split* graph G is a graph with two partitions, a stable set I and a clique K

such that $V(G) = I \bigcup K$. $E(G) \subseteq \{\{u, v\} | u \in I, v \in K\}$. For a chordal graph G and its tree decomposition T, a minimal vertex separator S, and an edge $e \in E(G)$, we consider fixed tree decompositions of $G \setminus S$ and $G.e$, denoted by $T \setminus S$ and $T.e$, respectively. $T \setminus S$ and $T.e$ are defined as follows: The vertex set of both $T \setminus S$ and $T.e$ are same as the vertex set of T. The removal of S and the contraction of e only changes the labels associated with the vertices. In $T \setminus S$, for each $x \in V(T \setminus S)$, we write $l(x) = l(x) \setminus S$, if $S \cap l(x) \neq \phi$, otherwise $l(x)$ is the same set as in T. In $T.e$, for each $x \in V(T.e)$, $l(x) = l(x) \setminus \{u, v\} \cup \{z_{uv}\}$, if $l(x) \cap \{u, v\} \neq \phi$. Otherwise, $l(x)$ is the same set as in T. Clearly, $T \setminus S$ and $T.e$ are tree decompositions of $G \setminus S$ and $G.e$, respectively.

2 The Structure of Contractible edges in *k*-connected Chordal Graphs

We first prove a theorem which characterizes the set of minimal vertex separators of a chordal graph. This result is used subsequently to prove our characterisation of contractible edges in chordal graphs.

Lemma 2. *Let G be a chordal graph and T be its tree decomposition. G is connected iff for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.*

Proof. Necessity: If G is connected then we need to show that for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$. We prove this by contradiction. Suppose there exists an edge $\{x, y\} \in E(T)$ and $l(x) \cap l(y) = \phi$. Consider the two components C_1 and C_2 obtained by removing the edge $\{x, y\}$. Assume that $x \in C_1$ and $y \in C_2$. Let $A = \bigcup_{z \in C_1} l(z)$, $B = \bigcup_{z \in C_2} l(z)$. Since T is a tree decomposition and $l(x) \cap l(y) = \phi$, it follows that $A \cap B = \phi$. Further, each edge $e \in E(G)$ is contained in the graph induced by A or B but not both. Hence G is disconnected. However, by our hypothesis G is connected. Hence our assumption is wrong. Therefore, if G is connected then for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.
Sufficiency: Given that for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$, we now show that G is connected. We show that $\forall u, \forall v \in V(G), u \neq v$, there exists a path between u and v in G. Let x, y be any two vertices in $V(T)$ such that $u \in l(x)$ and $v \in l(y)$. Consider the path $x = z_1, z_2, ..., z_j = y$ in the tree T. Further, $l(z_i) \cap l(z_{i+1})$ is non empty in T. This implies that there exists a vertex $r_i \in l(z_i) \cap l(z_{i+1})$. Hence the sequence of edges $\{u, r_1\}\{r_1, r_2\}...\{r_{j-1}, v\}$ is a uv path in G. The reason this is true in G is because G is a chordal graph and label of each node in T is a maximal clique. Therefore u and v are connected in G. Hence G is connected.

Note: For a simple graph G and any tree decomposition T, if G is connected then for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.

Theorem 1. *Let G be a k-connected chordal graph and let T be its tree decomposition. Let $M' = \{X | X = l(x) \cap l(y)$ where $\{x, y\} \in E(T)\}$ and $M'' = \{Y | Y \in M'$ and for all $Z \in M', Z \not\subset Y\}$. $M = M''$. In other words, M'' is the set of minimal vertex separators of G.*

Proof. $M'' \subseteq M$:
Let $S \in M''$. Clearly, for some $\{x,y\} \in E(T), S = l(x) \cap l(y)$. By definition, $T \setminus S$ is a tree decomposition of $G \setminus S$ and $l(x) \cap l(y) = \phi$. From Lemma 2 it follows that $G \setminus S$ is disconnected. Hence S is a vertex separator. Further, since $S \in M''$, it follows that there is no $S' \subset S$ such that $G \setminus S'$ is disconnected. Therefore, S is a minimal vertex separator.
$M \subseteq M''$
Let S be a minimal vertex separator(MVS). We now show that $S \in M''$. We now argue that there exist distinct $x, y \in V(T)$ such that $\{x,y\} \in E(T)$ and $l(x) \cap l(y) = S$. Since S is a MVS, $G \setminus S$ is disconnected. Since $G \setminus S$ is disconnected, in $T \setminus S$ there exists a pair of vertices x and y such that $l(x) \cap l(y) = \phi$. We now claim that in T, $S = l(x) \cap l(y)$. If $l(x) \cap l(y) \subset S$, this implies that S is not a MVS. However we are given the fact that S is a MVS. Therefore, $S = l(x) \cap l(y)$ and consequently $S \in M'$. Since S is a MVS it follows that $S \in M''$. Hence the proof.

Using the above characterization based on minimal vertex separators and tree decomposition of chordal graphs, we present the following theorem on the structure of contractible edges.

Theorem 2. *Let G be a k-connected chordal graph with $|V(G)| \geq (k+2)$. An edge $e = \{u,v\} \in E(G)$ is contractible iff one of the following holds*
(i) e is in a unique maximal clique in G
(ii) For $x, y \in V(T)$, $\{u,v\} \subset l(x) \cap l(y)$ and $\{x,y\} \in E(T)$, $|l(x) \cap l(y)| > k$.

Proof. Necessity:
(i): Given that e is contractible implies that $G.e$ is k-connected. If e is in a unique maximal clique in G, then we are done. In the case when e is not in a unique maximal clique, let $e \in l(x) \cap l(y)$ for some $\{x,y\} \in E(T)$. We now show that $|l(x) \cap l(y)| > k$. We prove this claim by contradiction. Let us assume that $|l(x) \cap l(y)| \leq k$. On contraction of e in G, the tree decomposition of $G.e$ is $T.e$. In $T.e$, the $|l(x) \cap l(y)| \leq k-1$. From lemma 2 it follows that that $l(x) \cap l(y)$ is a vertex separator of $G.e$, and since $|l(x) \cap l(y)| \leq k-1$, it follows that $G.e$ is $k-1$-connected. This is a contradiction to the hypothesis that $G.e$ is k-connected. Therefore, our assumption that $|l(x) \cap l(y)| \leq k$ is wrong. It follows that $|l(x) \cap l(y)| > k$.
Sufficiency: First, we consider the case when e is in a unique maximal clique and show that e is contractible. If e is in a unique maximal clique in G implies that e is contained in the label of a unique node in T. Therefore, for each $x, y \in T$, $|l(x) \cap l(y)|$ remains unchanged in $T.e$. From theorem 1 the connectivity of $G.e$ is at least as much as the connectivity of G. Therefore, e is contractible. In the case when $|l(x) \cap l(y)| > k$ for all $\{x,y\} \in E(T)$, after contracting e, in $T.e$ $|l(x) \cap l(y)|$ is at least k and hence the connectivity of $G.e$ is at least k, by theorem 1. Hence $G.e$ is k-connected. Therefore, e is contractible in G.

As a consequence of this lemma, it follows that each edge incident on a simplicial vertex in a k-connected chordal graph is contractible. Therefore, a k-connected

chordal graph has at least $2k$ contractible edges. We now apply the main lemma to understand contractible edges in split graphs. Let G be a non regular split graph. An edge $e = \{u, v\}$ such that $u \in K$ and $v \in I$ is contractible. Clearly such an edge e is in a unique maximal clique in G. By theorem 2 e is contractible. For the case when G is a regular k-connected split graph with at least $k + 2$ vertices, it follows that G is contraction critical, that is none of the edges of G are contractible. The reason is that, given that G is regular implies that there is exactly one vertex in I. Thus the resulting graph is a complete graph and each edge in every complete graph is non contractible. Therefore, G is contraction critical.

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